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Mini Review

Critical behavior and stability problem in a scalar field model

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Abstract

As shown in the works [1-3], the asymptotic behavior of the propagator in the Euclidean region of momenta for the model of a complex scalar field ϕ and a real scalar field χ with the interaction $g\phi^*\phi\chi$ drastically changes depending on the value of the coupling constant. For small values of the coupling, the propagator of the field ϕ behaves asymptotically as free, while in the strong-coupling region the propagator in the deep Euclidean region tends to be a constant.

In this paper, the influence of the vacuum stability problem of this model on this critical behavior is investigated. It is shown that within the framework of the approximations used, the addition of a stabilizing term of type C^∞ to the Lagrangian leads to a renormalization of the mass and does not change the main effect of changing the ultraviolet behavior of the propagator.

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Introduction

In this paper, we consider a problem of vacuum stability in a model of the N -component complex scalar field ϕ (phion) and real scalar matrix field χ (chion) with interaction $g\phi^*\phi\chi$ in Euclidean four dimensions. This model is the simplest model of the interaction of two fields and is often used as a prototype of more realistic theories to investigate the non-perturbative aspects in the quantum field theory (see, e.g., [4,5] and references therein).

The study of this model in the author's works [1-3] shows an interesting phenomenon – a change in the asymptotic behavior of the phion propagator in the deep Euclidean region at a certain critical value of the coupling. For small values of the coupling, the propagators behave as free, which is consistent with the widespread opinion about the dominance of perturbation theory for this super-renormalizable model. But the self-consistent solution for the propagator equation exists also for the strong coupling, where the asymptotic behavior changes dramatically – the phion propagator in the deep Euclidean region tends to some constant limits. The existence of this critical coupling looks like a specific ''phase transition`` for this four-dimensional field model.

For a more detailed investigation of this interesting phenomenon, it is necessary to study the stability of the ground state (vacuum) for this model. As it is well-known, since bosonic fields may be given arbitrarily large excitations, non-positively defined terms (type of $\phi^*\phi\chi$) in energy with such large field excitations will dominate the positive quadratic terms and destroy the ground state of such models [6].

As shown in the work of Gross, et al. [7], for this model, the problem can be solved by considering approximations with a finite number of chronic loops (it is to this type of approximation that the $1/N$ - expansion we use belongs).

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However, the proof of Gross, et al. is true only for the nonzero chion mass, whereas the solutions we have obtained for the phion propagator essentially exploit the masslessness of the phion and, consequently, the problem remains.

Another way to solve the stability problem is to add a positive definite fourth-order term to the Lagrangian. Of course, the question arises about the effect of such an additive on the results. The study of such a method of solving the problem of stability and the effect of an additive of the type ϕ^4 on the critical behavior of the propagator is the subject of the proposed work. To construct the effective action in the leading order of $1/N$-expansion and the equation for the propagator in this model, we use the bilocal source method[1].

The main result can be formulated very simply: the addition of such a stabilizing term in the leading order of the $1/N$ - expansion does not affect the effect of changing the asymptotic behavior of the propagator. The reason for this is that the addition of a stabilizing interaction to the equation for the phion propagator manifests itself only in an additional renormalization of the mass. Such renormalization has no effect on the asymptotic behavior of the propagator in the deep Euclidean region and, consequently, all the consequences of such behavior remain valid for the modified model with a stable vacuum.

Effective action and phion propagator

The Lagrangian of the model is

$$L = -\partial_\mu \phi_a^* \partial_\mu \phi_a - m^2 \phi_a^* \phi_a - \frac{1}{2}(\partial_\mu \chi_{ab})^2 + \frac{g}{\sqrt{N}} \phi_a^* \phi_b \chi_{ab}, \quad (1)$$

where $a,b=1,\dots,N$. The usual agreement on the summation of repeated discrete indices is implied. It is convenient to construct a $1/N$ -expansion for this vector-matrix model in the formalism of the Schwinger-Dyson equations[2] using a matrix bilocal source $\eta_{ba}(y,x)$ of a complex field ϕ_a. Generating functional G correlation functions (vacuum averages) is a functional integral over the fields, and the derivative of G the over bilocal source is a two-point function

$$\delta G / \delta \eta_{ba}(y,x) = -<\phi_a(x)\phi_b^*(y)>.$$

The translational invariance of the functional integration measure leads to Schwinger-Dyson equations for generating functional G. Then excluding the chion field and the using Bose-symmetry condition, we obtain for generating functional

$$Z = \log G$$

the equation

$$(m^2 - \partial_x^2)\frac{\delta Z}{\delta \eta_{ba}(y,x)} + \int dx_1 \eta_{aa'}(x,x_1)\frac{\delta Z}{\delta \eta_{ba'}(y,x_1)} + \delta_{ab}\delta(x-y)$$

$$= -\frac{g^2}{N}\int dx_1 D_c(x-x_1)\Big[\frac{\delta^2 Z}{\delta \eta_{ba}(y,x_1)\delta \eta_{b'b'}(x_1,x)} + \frac{\delta Z}{\delta \eta_{b'b'}(x_1,x)}\frac{\delta Z}{\delta \eta_{ba}(y,x_1)}\Big]. \quad (2)$$

Here $D_c = -\partial^{-2}$.

To construct the $1/N$ -expansion of this equation, it is convenient to use the Legendre transform of the generating function $Z[\eta]$. Consider the definition of a propagator with a source η

$$\frac{\delta Z}{\delta \eta_{ba}(y,x)} = -\Delta_{ab}(x,y|\eta) \quad (3)$$

as a functional equation for $\eta = \eta[\Delta]$ and resolving this equation, we can define a new generating functional $\Gamma[\Delta]$ (effective action) of a new functional variable Δ:

$$\Gamma[\Delta] = Z + \int dx_1 dy_1 \Delta_{ab}(x_1,y_1)\eta_{ba}(y_1,x_1). \quad (4)$$

It follows from this definition that $\delta\Gamma / \delta\Delta_{ba}(y,x) = \eta_{ab}(x,y|\Delta)$. Legendre-transformed SDE (2) can be rewritten as:

$$\frac{\delta \Gamma}{\delta \Delta_{ba}(y,x)} = (\Delta^{-1})_{ab}(x,y) - (m^2 - \partial^2)\delta_{ab}\delta(x-y) + \frac{g^2}{N}\delta_{ab} D_c(x-y)\, tr\, \Delta(x,y) -$$

$$-\frac{g^2}{N}\int dx_1 dy_1 D_c(x-x_1)\frac{\delta \Delta_{aa'}(x_1,y_1)}{\delta \eta_{b'b'}(x_1,x)}(\Delta^{-1})_{a'b}(y_1,y). \quad (5)$$

(In this equation $\delta\Delta / \delta\eta$ should be expressed as a function of Δ).

It can be shown (see [3]) that the last term of equation (5) has a higher order $1/N$ and the leading approximation equation for generating functional Γ is

$$\frac{\delta \Gamma_0}{\delta \Delta_{ba}(y,x)} = (\Delta^{-1})_{ab}(x,y) - \delta_{ab}(m^2 - \partial^2)\delta(x-y) + \frac{g^2}{N}\delta_{ab} D_c(x-y)\, tr\, \Delta(x,y). \quad (6)$$

Correspondingly, the leading-order equation for the phion propagator $\Delta_{ab}(p) = \delta_{ab}\Delta(p^2)$ in the momentum space is

$$\Delta^{-1}(p^2) = m^2 + p^2 - g^2 \int \frac{d^4 q}{(2\pi)^4}\frac{\Delta(q^2)}{(p-q)^2}. \quad (7)$$

To renormalize this equation one should add counter-terms of renormalization of the phion mass δm^2 and the phion-field renormalization z. We shall use the normalization at zero momentum

$$\Delta^{-1}(0) = m^2, \frac{d\Delta^{-1}}{dp^2}\Big|_{p^2=0} = 1. \quad (8)$$

Here and below Δ and m are the renormalized quantities.

The renormalized equation for the phion propagator becomes

$$\Delta^{-1}(p^2) = m^2 + p^2 + \Sigma_r(p^2), \quad (9)$$

[1] A formalism of multilocal sources was elaborated in the framework of quantum statistics by De Dominisis and Martin [8] and elaborated in quantum field theory in works [9-11] (see also [12] for the modern applications).

[2] See also [3] for details.

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where $\Sigma_r(p^2) = \Sigma(p^2) - \Sigma(0) - p^2\Sigma'(0)$ and

$$\Sigma(p^2) = -g^2 \int \frac{d^4q}{(2\pi)^4} \frac{\Delta(q^2)}{(p-q)^2}. \quad (10)$$

After the angle integration we obtain for the phion propagator the integral equation:

$$\Delta^{-1}(p^2) = m^2 + (1-\kappa)p^2 + 2\kappa m^2 \int_0^{p^2} \Delta(q^2)(1 - \frac{q^2}{p^2})dq^2, \quad (11)$$

where

$$\kappa \equiv \frac{g^2}{32\pi^2 m^2} \quad (12)$$

is dimensionless coupling.

This integral equation can be reduced to the non-linear differential equation

$$\frac{d^2}{(dp^2)^2}(p^2\Delta^{-1}(p^2)) = 2(1-\kappa) + 2\kappa m^2 \Delta(p^2). \quad (13)$$

Depending on the value k, three different types of positive solutions[3] are possible.

In the weak-coupling region, $k<1$ the asymptotic solution at large p^2 is

$$\Delta^{-1}(p) = (1-\kappa)p^2. \quad (14)$$

This asymptotic solution is positive $k<1$ and corresponds to the asymptotically-free behavior of the propagator.

At critical coupling $k=1$ the propagator at large momenta is

$$\Delta_{cr} = \frac{1}{m}\sqrt{\frac{3}{8p^2}}. \quad (15)$$

and drastically differs from the asymptotically-free behavior in the weak-coupling region.

At $k>1$ differential equation (13) has the positive exact solution

$$\Delta_s = \frac{\kappa - 1}{\kappa} \frac{1}{m^2}. \quad (16)$$

It can be proven (see [2]) that Δ_s is the asymptotic solution of the integral equation (11) at $p^2 \to \infty$.

A change of asymptotic behavior in the vicinity of the critical value of coupling is similar to the re-arrangement of a physical vacuum in the strong external field. This ''fall on the center`` is related to the term $1/r^2$ in a potential (see [13] and references therein). The potential U, which corresponds to the propagator, is defined as the response to a static source $j(x) = \delta^3(x)$:

$$U(r) \sim -\frac{g^2}{m^2} \int dx_1 \Delta_M(x - x_1) j(x_1), \quad (17)$$

[3]Non-positive solutions necessarily contain Landau singularities and are therefore physically unacceptable.

Where Δ_M is the propagator in pseudoeuclidean Minkowski space (see, e.g., [14])? At critical value $k=1$ for propagator (15), we obtain

$$U(r) \sim -\frac{1}{m}\frac{1}{r^2}. \quad (18)$$

It is really a potential of ``fall on the center''. Despite all the obvious limitations of this analogy, it undoubtedly indicates the related nature of these phenomena.

The phion propagator in the strong-coupling region asymptotically approaches a constant. It is not something unexpected if we remember the well-known conception of the static ultra-local approximation, or ''static ultralocal model``, which can be considered as a starting point for the strong-coupling expansion (see [15] and references therein). In contrast to the ultra-local approximation, our solution has the standard pole behavior for the small momenta.

Stabilization

As noted above, the scalar field model we are considering does not have a stable ground state and therefore cannot serve as a basis for realistic physical models of particle interactions. The reason why there is no ground state is as follows: bosonic fields can experience arbitrarily large excitations, and a non-positive cubic term in energy with such large field excitations will dominate the positive quadratic terms [6].

As shown in the work of Gross, et al. [7], the situation can be corrected if we restrict ourselves to an approximation containing a finite number of phionic loops (of course, the 1-N-expansion is such an approximation). The theory is stable if the Fock space of all intermediate states is bounded by a finite number of phion loops. However, the proof of this fact in the work [7] is essentially based on the massiveness of the chion field, while all the results we obtained for the phion propagator are essentially based on the masslessness of the chion. In this regard, the problem of the stability of the ground state in this model becomes relevant again.

We show that it is possible to modify the model under consideration corresponding to a stable vacuum and including, on the other hand, all the above effects associated with a change in the asymptotic behavior of the phion propagator.

As is known (see [6]), the presence of positive definite structures of the fourth degree in the Lagrangian, such as $\lambda\phi^4$ with an arbitrarily small positive coupling constant λ guarantees the existence of the ground state. We shall prove that adding the such stabilizing term

$$-\frac{\lambda}{N}(\phi_a^*\phi_a)^2 \quad (19)$$

to the original Lagrangian (1) does not affect the main effect – the change of asymptotic behavior of the phion propagator. Really, adding this term to the Lagrangian leads to a modification of the equations (2), (5) and (6) by replacing

$$g^2 D_c(x-y) \Rightarrow g^2 D_c(x-y) - \lambda\delta(x-y). \quad (20)$$

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Consequently, an additional term $\Gamma_{\lambda} = -\frac{\lambda}{2N}(Tr\,\Delta)^2$ will appear in the expression for effective action and a corresponding positively-definite stabilizing additive will appear in the vacuum energy density.

At the same time, the non-renormalized equation for the phion propagator will differ from what we have studied only by replacing

$$m^2 \Rightarrow m^2 + \lambda\Delta|_{x\to 0},$$

i.e., by redefinition of the counter-term of mass renormalization. Such redefinition leads to the same renormalized equations (9) – (11) and it will have no effect on the results of the investigation of the phion propagator. Hence all conclusions remain in full force.

Conclusion

The obtained result shows that the phenomenon of changing asymptotic behavior does not contradict the existence of a stable ground state of strongly interacting boson fields.

This critical phenomenon is a quantum-field analog of the re-arrangement of the physical vacuum in a strong external field. Further study of similar phenomena in more realistic models of strongly interacting bosons (including glue) is of undoubted interest.

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