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Research Article

Certain results of Aleph-Function based on natural transform of fractional order

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Abstract

In this research article, a new type of fractional integral transform namely the N-transform of fractional order is proposed, and derived a number of useful results of a more generalized function (Aleph-function) of fractional calculus by making use of the N-transform of fractional order. Further, the relation between it and other fractional transforms is given and some special cases have also been discussed.

Introduction

Our translation of real-world problems to mathematical expressions relies on calculus, which in turn relies on the differentiation and integration operations of arbitrary order with a sort of misnomer fractional calculus which is also a natural generalization of calculus and its mathematical history is equally long. It plays a significant role in a number of fields such as physics, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, elasticity, control theory, engineering mathematics, and many others. Fractional calculus like many other mathematical disciplines and ideas has its origin in the quest of researchers to expand its applications to new fields. This freedom of order opens new dimensions and many problems of applied sciences can be tackled in a more efficient way by means of fractional calculus.

Laplace and Sumudu transformations are closely linked to natural transformation. The Natural transform, also known as the N-transform, was initially introduced by Khan and Khan (2008) [1]; Al-Omari (2013) [2]; Belgacem and Silambarasan [3] explored its features (2012b). Maxwell's equations were solved using the Natural transform in Belgacem and Silambarasan (2011, 2012a) [4]. Belgacem and Silambarasan's (2011, 2012c) works on Natural transformation can be found here [5] for more information. If we assume that the function is fractional derivative and continuous, the Natural transform often works with continuous and continuously differentiable functions. The Natural transform, like the Laplace and Sumudu transforms, does not work since the function is not derivative. In a similar vein, we must establish a new term that we will call fractional Natural transform [6-8].

The purpose of this research article is to calculate the fractional order natural transform of the Aleph function.

Definitions and preliminaries used in this paper

Classical Laplace transform: The Laplace transform is very useful in analysis and design for systems that are linear and time-invariant (LTI). Beginning in about 1910, transform techniques were applied to signal processing at Bell Labs for signal filtering and telephone long-line communication by

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H. Bode and others. Transform theory subsequently provided the backbone of Classical Control Theory as practiced during the World Wars and up to about 1960, when State Variable techniques began to be used for control design. Pierre Simon Laplace was a French mathematician who lived from 1749-1827, during the Age of Enlightenment characterized by the French Revolution, Rousseau, Voltaire and Napoleon Bonaparte.

Suppose f(t) is a real-valued function defined over the interval (0,∞). The Laplace transform of *f(t)* is defined by

$$L\left[f\left(t\right)\right]=\int_0^\infty e^{-st}f\left(t\right)d(t)$$

Or $f\left(s\right)=\int_0^\infty e^{-st}f\left(t\right)d(t)$

The Laplace transform is said to exist if the above integral is convergent for some values of s.

The Inverse Laplace Transform can be described as the transformation into a function of time. In the Laplace inverse formula, *f(s)* is the Transform of *f(t)*, while in the Inverse Transform *f(t)* is the Inverse Laplace Transform of *f(s)*. Therefore, we can write this Inverse Laplace transform formula as follows:

$$f\left(t\right)=L^{-1}\{f\}\left(t\right)=\frac{1}{2\pi i}\lim_{L\to\infty}\int_{\gamma-iL}^{\gamma+iL}e^{st}f(s)ds$$

Natural transform: In mathematics, the Natural transform is an integral transform similar to the Laplace transform and Sumudu transform, introduced by Zafar Hayat Khan[1] 2008. It converges to both Laplace and Sumudu transform just by changing variables. Given the convergence of the Laplace and Sumudu transforms, the N-transform inherits all the applied aspects of both transforms. Most recently, F. B. M. Belgacem [2] has renamed it the natural transform and has proposed a detailed theory and applications. The natural transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $R(u, s)$, defined by:

$$R\left(u,s\right)=N\left[f\left(t\right)\right]=\int_0^\infty e^{-st}f\left(ut\right)dt, Re(S)>0, u(-\tau_1,\tau_2) \quad (1)$$

Provided the function $f(t) \in R^2$ *is defined in the set*

$$A=\{f\left(t\right)\ |\exists M,\tau_1,\tau_2>0.|f(t)|<Me^{\frac{|t|}{\tau_j}}$$

Khan [1] showed that the above integral converges to the Laplace transform when $u = 1$ and into Sumudu transform for $s = 1$.

Laplace transform of error function:

$$L\{\mathrm{erf}(t)\}=\frac{1}{s}\exp\left(\frac{s^2}{4}\right)erfc\left(\frac{s}{2}\right)$$

Where: L{f} denotes the Laplace transform of the function f.

erf denotes the error function.

erfc denotes the complementary error function.

Fractional natural transform of order α :

$$R_\alpha^+(u,s)=N_\alpha^+\left[f\left(x\right)\right]=\int_0^\infty E_\alpha(-s^\alpha x^\alpha)f\left(ux\right)(dx)^\alpha, 0<\alpha\leq 1.$$

Or

$$R_\alpha^+(u,s)=\lim_{M\to\infty}\int_0^M E_\alpha(-s^\alpha x^\alpha)f\left(ux\right)(dx)^\alpha \quad (2)$$

Where s, u ∊ C and $E_\alpha(x)$ is the Mittag–Leffler function, $E_\alpha(z)=\sum_{n=0}^\infty \frac{x^n}{\alpha n!}$

Fractional Laplace transform reported in Jumarie (2009a) [9]:

From the above definition, when u = 1

$$L_\alpha^+(1,s)L_\alpha^+\left[f\left(x\right)\right]=\int_0^\infty E_\alpha(-s^\alpha x^\alpha)f\left(x\right)(dx)^\alpha,\ 0<\alpha\leq 1.$$

Or

$$L_\alpha^+(1,s)==\lim_{M\to\infty}\int_0^M E_\alpha(-s^\alpha x^\alpha)f\left(x\right)(dx)^\alpha \quad (3)$$

Where s ∊ C and $E_\alpha(x)$ is the Mittag–Leffler function, $E_\alpha(z)=\sum_{n=0}^\infty \frac{x^n}{\alpha n!}$

Fractional Sumudu transform which is proposed by Gupta, Sharma and Kiliçman (2010):

From the above definition, when S = 1

$$S_\alpha^+(u,1)=S_\alpha^+\left[f\left(x\right)\right]=\int_0^\infty E_\alpha(-x^\alpha)f\left(ux\right)(dx)^\alpha, 0<\alpha\leq 1.$$

Or

$$S_\alpha^+(u,1)==\lim_{M\to\infty}\int_0^M E_\alpha(-x^\alpha)f\left(ux\right)(dx)^\alpha \quad (4)$$

Where u ∊ C and $E_\alpha(x)$ is the Mittag–Leffler function, $E_\alpha(z)=\sum_{n=0}^\infty \frac{x^n}{\alpha n!}$

Aleph-function:

The Aleph function is defined in terms of the Mellin-Barnes type integral in the following manner [10]:

$$\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]$$

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$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}s]}z^s ds \quad (5)$$

Lemma-I: For instance the fractional natural transform of the $f(x)=x^{n\alpha}$, $n\in N$ then

$$N_\alpha^+\left[x^{n\alpha}\right]=\int_0^\infty E_\alpha(-s^\alpha x^\alpha)(ux)^{n\alpha}(dx)^\alpha =$$

$$u^{n\alpha}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)(x)^{n\alpha}(dx)^\alpha$$

We put $t = xs$ we get

$$N_\alpha^+\left[x^{n\alpha}\right]=\frac{u^{n\alpha}}{s^{(n+1)\alpha}}\int_0^\infty E_\alpha(-t^\alpha)(t)^{n\alpha}(dt)^\alpha$$

Or

$$N_\alpha^+\left[x^{n\alpha}\right]=\frac{(\alpha!)u^{n\alpha}}{s^{(n+1)\alpha}}\Gamma_{\acute{a}}\left(n+1\right)$$

Note: $\Gamma_{\acute{a}}\left(n\right)=\frac{1}{(\alpha!)}\int_0^\infty E_\alpha(-x^\alpha)(x)^{(n-1)\alpha}(dx)^\alpha$

Lemma-II: For instance the fractional Laplace transform of the $f(x)=x^{n\alpha}$, $n\in N$ then

$$L_\alpha^+\left[x^{n\alpha}\right]=\int_0^\infty E_\alpha(-s^\alpha x^\alpha)(x)^{n\alpha}(dx)^\alpha=\int_0^\infty E_\alpha(-s^\alpha x^\alpha)(x)^{n\alpha}(dx)^\alpha$$

We put $t = xs$. we get

$$L_\alpha^+\left[x^{n\alpha}\right]=\frac{1}{s^{(n+1)\alpha}}\int_0^\infty E_\alpha(-t^\alpha)(t)^{n\alpha}(dt)^\alpha$$

Or

$$L_\alpha^+\left[x^{n\alpha}\right]=\frac{(\alpha!)}{s^{(n+1)\alpha}}\Gamma_{\acute{a}}\left(n+1\right)$$

Note: $\Gamma_{\acute{a}}\left(n\right)=\frac{1}{(\alpha!)}\int_0^\infty E_\alpha(-x^\alpha)(x)^{(n-1)\alpha}(dx)^\alpha$

Lemma-III: For instance the fractional Sumudu transform of the $f(x)=x^{n\alpha}$, $n\in N$ then

$$S_\alpha^+\left[x^{n\alpha}\right]=\int_0^\infty E_\alpha(-x^\alpha)(ux)^{n\alpha}(dx)^\alpha =$$

$$u^{n\alpha}\int_0^\infty E_\alpha(-x^\alpha)(x)^{n\alpha}(dx)^\alpha$$

We put $t = x$, we get

$$S_\alpha^+\left[x^{n\alpha}\right]=u^{n\alpha}\int_0^\infty E_\alpha(-t^\alpha)(t)^{n\alpha}(dt)^\alpha$$

Or

$$S_\alpha^+\left[x^{n\alpha}\right]=(\alpha!)u^{n\alpha}\Gamma_{\acute{a}}\left(n+1\right)$$

Note: $\Gamma_{\acute{a}}\left(n\right)=\frac{1}{(\alpha!)}\int_0^\infty E_\alpha(-x^\alpha)(x)^{(n-1)\alpha}(dx)^\alpha$

Main results

Fractional natural transform of order α

In this section, we derived the fractional natural transform of order α in relationship with the known generalized function of fractional calculus known as the Aleph function.

Theorem (1): Let $N_\alpha^+\left[f(x)\right], 0<\alpha\le 1$, be the fractional natural transform of order α associated with Aleph-function. Then there holds the following relationship

$$N_\alpha^+\left\{\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}\right\}=$$

$$\frac{1}{s}\aleph_{p_i,q_i;\tau_i;r}^{m,n+1}\left[\frac{u}{s}\left|\begin{matrix}(0,1)\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]$$

Provided the function $f(t)\in R^2$

Proof: By using the definition of the generalized function of fractional Aleph -function and fractional natural transform of order α we get

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$N_\alpha^+\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}z^k dk\right\};\ \mathrm{Re}(\alpha)>0$$

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\right\}N_\alpha^+\left\{z^k\right\}$$

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\right\}N_\alpha^+\left\{z^k\right\}$$

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By making use of lemma –I in above equation, we get

$$N_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\frac{u^k}{s^{(k+1)}}\Gamma(k+1)$$

Or

$$N_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{s}\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)\Gamma(1-0+k)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\frac{u^k}{s^k}$$

Or

$$N_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{s}\aleph^{m,n+1}_{p_i,q_i;\tau_i;r}\left[\frac{u}{s}\left|\begin{matrix}(0,1)(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]$$

This completes the proof of the theorem.

Fractional laplace transform of order α

In this section, we derived the fractional Laplace transform of order α in relationship with the known function of fractional calculus known as the Aleph function.

Theorem (2): Let $L_\alpha^+\left[f(x)\right], 0<\alpha\le 1$, be the fractional Laplace transform of order α associated with Aleph-function. Then there holds the following relationship

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{s}\aleph^{m,n+1}_{p_i,q_i;\tau_i;r}\left[s^{-1}\left|\begin{matrix}(1,0)(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]$$

Provided the function $f(t)\in R^2$

Proof: By using the definition of the generalized function of fractional ML-function and fractional Laplace transform of order α we get

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$L_\alpha^+\left\{\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}z^k\mathrm{dk}\right\};\ \mathrm{Re}(\alpha)>0$$

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\ L_\alpha^+\{z^k\}$$

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\ L_\alpha^+\{z^k\}$$

By making use of lemma –II in the above equation, we get

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\frac{1}{s^{(k+1)}}\Gamma(k+1)$$

Or

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)\Gamma(k+1)\Gamma(1-0+k)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\frac{1}{s^{(k+1)}}\mathrm{dk}$$

$$L_\alpha^+\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_j,B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

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$$\frac{1}{s}\aleph^{m,n+1}_{p_i,q_i;\tau_i;r}\left[s^{-1}\left|\begin{matrix}(1,0)\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]$$

This completes the proof of the theorem.

Fractional Sumudu transform of order α

In this section, we derived the fractional Sumudu transform of order α in relationship with the known function of fractional calculus known as ML-function.

Theorem (3): Let $S^{+}_{\alpha}\left[f(x)\right], 0<\alpha\le 1$, be the fractional Sumudu transform of order α associated with Aleph-function. Then there holds the following relationship

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=\frac{1}{s}E^{1}_{a}\left(\frac{u}{s}\right)$$

Provided the function $f(t)\in R^2$

Proof: By using the definition of the generalized function of fractional Aleph -function and fractional Sumudu transform of order α we get

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$S^{+}_{\alpha}\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}z^{k}\,dk;\ \mathrm{Re}(\alpha)>0$$

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\,S^{+}_{\alpha}\{z^{k}\}$$

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\,S^{+}_{\alpha}\{z^{k}\}$$

By making use of lemma –III in the above equation, we get

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}dk\,u^{k}\Gamma(k+1)$$

Or

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_jk\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_jk\right)\Gamma(1-0+k)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}u^{k}dk$$

$$S^{+}_{\alpha}\left\{\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[Z\left|\begin{matrix}\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$

$$\aleph^{m,n+1}_{p_i,q_i;\tau_i;r}\left[u\left|\begin{matrix}(0,1)\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m}\left[\tau_i\left(b_j,B_{ji}\right)\right]_{m+1,q_i}\end{matrix}\right.\right]$$

This completes the proof of the theorem.

Special cases

In this section, we discuss some of the important special cases of the main results established discussed above, If we take ∞ = τ_i =1 in the theorems (1), (2), and (3), we get well-known results of ordinary calculus like the Natural transform of Saxena's I-function, Laplace transform of Saxena's I-function, and finally ordinary Sumudu transform of Saxena's I-function as reported in [11].

Conclusion

We are trying for more specified and detailed results of this transformation. The results proved in this paper give some contributions to the theory of the fractional order transform, believed to be new and are likely to find certain applications to the solution of the fractional differential and integral equations order equations. The importance of this work is to find the half derivative, fractional order Laplace Natural transforms, etc.

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